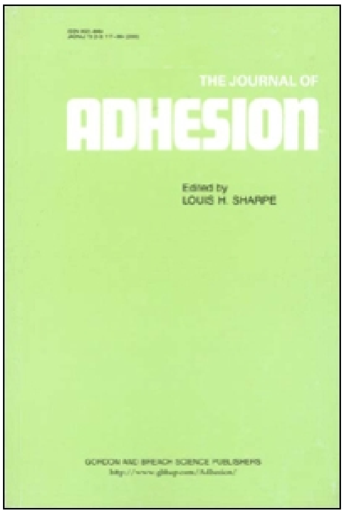


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# The Influence of Variations in Glue-Line Thickness on the Load Transfer in Double Overlap Joints

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The equilibrium equation of the classical theory of joints is not, in general, soluble in closed form for variable glue-line thickness. It is shown, however, that analytical results describing the influence of thickness variations can be obtained by two methods: a perturbation method and an inverse method. Detailed calculations are given for thickness variations which are most pronounced at one end of the joint and which can be characterized by two parameters, namely an amplitude and a decay length. Of the two methods, the perturbation method is the more flexible in application, but its accuracy relies on the amplitude being sufficiently small, while the inverse method can lead to an exact solution which can be used to assess the range of accuracy of the two-term perturbation expansion. Certain practical implications of the results obtained are discussed.

## INTRODUCTION

The classical theory of bonded overlap joints treats the adherends as one-dimensional continua and thus reduces the equations of equilibrium to ordinary differential equations.[1] In a comprehensive study of bonded joints, Hart-Smith[2] has highlighted the usefulness of this theory in providing basic insights as well as leading to practical design charts. More recently,[3] he has discussed briefly the influence of variations in glue-line thickness, using numerical results obtained for two particular cases of thickness variation in the basic configuration of the double overlap joint. While these numerical results give an intuitive indication of what can be expected in other cases, it would clearly be desirable to have explicit formulae for assessing the influence of thickness variations on the distribution of adhesive shear stress. The aim of this note is to show how such formulae can be derived. Two methods will be used: a perturbation method and an inverse method.

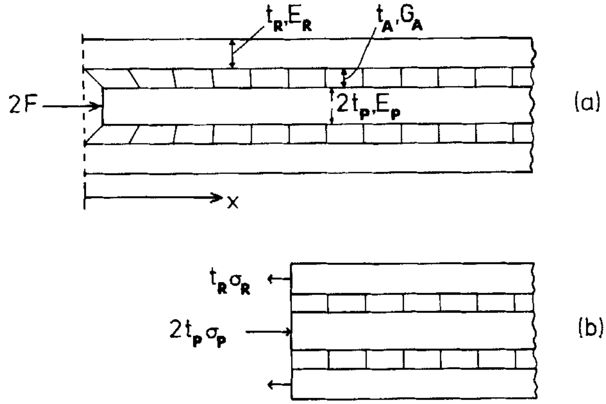


FIGURE 1 (a) The double overlap joint configuration being considered. (b) Free-body diagram for deriving Eq. (6).



The overlap joint has received renewed interest in the context of cracked plates repaired by bonded reinforcements. It has been shown[4] that an upper bound for the crack extension force in the repaired plate can be derived by considering the work done by the applied force in the overlap joint configuration of Figure 1(a). We shall not elaborate further on this connection, which is discussed in detail in,[4,5] except to note that the identifying subscripts and the loading configuration chosen in Figure 1(a) follow from that repair context. Thus $E_P$, $E_R$ denote the Young's moduli, $2t_P$, $t_R$ the thicknesses of the inner and outer adherends respectively (the *plate* and the *reinforcements* in the repair context), while $G_A$, $t_A$ denote the shear modulus and thickness of each adhesive layer. The methods which will be described, however, can readily be applied to other configurations.

The principal feature of the classical theory of joints is the exponential decay of the adhesive shear stress from the ends of the joint. Consequently one can identify a load-transfer length at each end and the joint strength becomes practically independent of the length of overlap when the overlap is longer than the sum of these transfer lengths, provided that the joint strength is determined by failure of the adhesive, which we shall assume to be the case throughout the following analysis. Since the aim of this analysis is to show how the classical results are altered by a variable glue-line thickness, we shall assume that the overlap is sufficiently long for the interaction between the load-transfer regions at the ends to be negligible, so that we can focus attention on the load-transfer characteristics at only one end of a joint, as in Figure 1(a). It will also be assumed that the adhesive as well as the adherends deform elastically, so that the stress distributions for the more conventional loading configuration, in which no external force is applied to the inner adherend at $x = 0$, could be derived from the present results by linear superposition; in fact, it can be readily verified that the adhesive shear stress near the end $x = 0$ is the same for these two cases, so that the present analysis gives directly the amount of load which can be transferred at that end by an elastic-brittle adhesive.

## THE DIFFERENTIAL EQUATION FOR VARIABLE GLUE-LINE THICKNESS

The classical theory of bonded joints is based on the following assumptions:

i) Each adherend is treated as a one-dimensional continuum whose deformation is specified by a longitudinal displacement $u$ and a longitudinal normal stress $\sigma$. The stress-displacement relations are

$$\sigma_P(x) = E_P u_P'(x), \qquad \sigma_R(x) = E_R u_R'(x), \tag{1}$$



where the dash denotes a differentiation with respect to $x$.

ii) Each adhesive layer acts as a shear spring with the adhesive shear stress $\tau_A$ given by

$$\tau_A(x) = \{G_A/t_A\}\{u_P(x) - u_R(x)\}. \tag{2}$$

In the classical theory, the glue-line thickness is assumed to be constant. To retain an easy correspondence with that theory, we shall write the actual thickness $t_A(x)$ in the form

$$t_A(x) = t_A h(x), \tag{3}$$

so that $t_A$ on the right hand side (and hereafter) is again a constant, which can be considered to be the *nominal thickness*, while $h(x)$ describes the normalized thickness variation. Thus for our case Eq. (2) can be written as follows,

$$h(x)\tau_A(x) = (G_A/t_A)\{u_P(x) - u_R(x)\}. \tag{4}$$

We note that this form of the equation could equally be used to describe a variation in the adhesive shear modulus, or, more generally, to describe a variation in the ratio of shear modulus to thickness of the adhesive layer.

iii) The shear tractions exerted by the adhesive can be replaced by an equivalent body force distributed uniformly across the thickness of each adherend, leading to the following differential equations of equilibrium:

$$t_P \sigma_P'(x) = -t_R \sigma_R'(x) = \tau_A(x). \tag{5}$$

The traditional approach[1] now is to derive a differential equation for $\tau_A$, but it is more convenient in the present context to derive one for $\sigma_P$ instead. To that end, consider the free-body diagram shown in Figure 1(b). As there is no external force applied at $x \to \infty$, the condition of equilibrium leads to the relation

$$t_P\sigma_P(x)+t_R\sigma_R(x) = 0. \tag{6}$$

Now, by differentiating both sides of Eq. (4) with respect to $x$, using (5) to replace $\tau_A$ by $\sigma'_P$ and using Eqs. (1), (6) to express $\{u'_P(x)-u'_R(x)\}$ in terms of $\sigma_P$, we obtain the following basic equation,

$$\{h(x)\sigma'_P(x)\}'-\beta^2\sigma_P(x) = 0, \tag{7}$$

where

$$\beta^2 = (G_A/t_A)\{(E_P t_P)^{-1}+(E_R t_R)^{-1}\}. \tag{8}$$

This equation must now be solved subject to the boundary conditions

$$t_P\sigma_P(x=0) = -F, \tag{9}$$

$$\sigma_P(x \to \infty) = 0. \tag{10}$$

$2F$ is the applied load per unit width, the width being measured perpendicular to the cross-section shown in Figure 1(a).

Despite its simple appearance, Eq. (7) is not in general soluble in closed form. To proceed we shall first use a perturbation method[6] and, secondly, an inverse method in which $\sigma_P(x)$ is specified and the corresponding $h(x)$ is then determined—a simple but effective device which can profitably be used in many contexts.[7]



## PERTURBATION METHOD

Let

$$h(x) = 1+\varepsilon f(x), \tag{11}$$

where $\varepsilon(\ll 1)$ is the small perturbation parameter. In principle, $f(x)$ can be any arbitrary function having a bounded second derivative, but it will prove to be analytically convenient to choose

$$f(x) = \pm\exp(-\kappa x), \; \kappa > 0. \tag{12}$$

This choice gives a thickness variation which is most pronounced at $x = 0$ and which has a characteristic length of $1/\kappa$. With the minus sign in Eq. (12) we would have a constriction (or pinch off[3]) of the adhesive layer, which could arise in practice from a beading of the inner adherend at its end $x = 0$. It is intuitively evident that such beading will increase the maximum adhesive shear stress for a given load, while an increase in thickness (corresponding to the choice of the plus sign in Eq. (12)) would decrease the maximum shear stress. The object of the analysis is to derive explicit quantitative estimates for these expectations.

In accordance with the standard perturbation procedure we can now write

$$\sigma_P(x) = \sigma_P^0(x) + \varepsilon\sigma_P^1(x) + \mathcal{O}(\varepsilon^2), \tag{13}$$

where $\varepsilon\sigma_P^1$ represents the first-order correction to the zero-order solution $\sigma_P^0$, and $\mathcal{O}(\varepsilon^2)$ stands for terms of order $\varepsilon^2$ or smaller. Since we are dealing with a linear problem, the other dependent variables admit similar asymptotic representations for small $\varepsilon$, in particular

$$\tau_A(x) = \tau_A^0(x) + \varepsilon\tau_A^1(x) + \mathcal{O}(\varepsilon^2). \tag{14}$$

## The zero-order approximation

By setting $\varepsilon = 0$ in (11) we recover the equation of the classical theory (corresponding to constant glue-line thickness $t_A$) for the zero-order approximation:

$$(\sigma_P^0)'' - \beta^2\sigma_P^0 = 0, \tag{15}$$

subject to the boundary conditions

$$t\sigma_P^0(x=0) = -F, \qquad \sigma_P^0(x \to \infty) = 0. \tag{16}$$

The appropriate solution is readily found to be

$$\sigma_P^0(x) = -(F/t_P)\exp(-\beta x). \tag{17}$$

This shows the exponential pick-up of load at the ends of a joint, according to the classical theory, the load-transfer length being $1/\beta$.

From Eq. (5) the corresponding adhesive shear stress is

$$\tau_A^0(x) = F\beta\exp(-\beta x). \tag{18}$$

Thus, for the case of uniform glue-line thickness, the maximum applied load $F_m^0$ which can be sustained by an elastic-brittle adhesive with a failure stress $\tau_A^f$ is given by

$$F_m^0 = \tau_A^f/\beta. \tag{19}$$

## The first-order approximation

Substituting Eqs. (11) and (13) in Eq. (7) and retaining only the terms of order $\varepsilon$ we derive the following equation for the first order correction:

$$(\sigma_P^1)'' - \beta^2\sigma_P^1 = -\{f(x)(\sigma_P^0)'\}', \tag{20}$$

subject to the boundary conditions

$$\sigma_P^1(x = 0) = \sigma_P^1(x \to \infty) = 0. \tag{21}$$

A particular integral of this equation can be found in a convenient analytical form if $f(x)$ is the exponential function in Eq. (12) with $\kappa \neq \beta$. Then one can readily verify that the appropriate solution is

$$\sigma_P^1(x) = \mp(F/t_P)\{\beta(\beta+\kappa)/\kappa(2\beta+\kappa)\}(1-e^{-\kappa x})e^{-\beta x}. \tag{22}$$

The corresponding $\tau_A^1$ is obtained by differentiation, and, collecting the results, we find that a glue-line thickness

$$t_A(1 \pm \varepsilon e^{-\kappa x}), \quad \varepsilon \ll 1, \kappa > 0, \tag{23}$$

leads to the following adhesive shear stress:

$$\tau_A(x) = F\beta e^{-\beta x}\left[1 \pm \varepsilon \frac{(\beta+\kappa)}{\kappa(2\beta+\kappa)}\{\beta-(\beta+\kappa)e^{-\kappa x}\}\right] + \mathcal{O}(\varepsilon^2). \tag{24}$$



## Discussion

With the choice of the minus sign in Eq. (23), which corresponds to a constriction of the adhesive layer, the maximum value of $\tau_A$ occurs at $x = 0$. For an elastic-brittle adhesive with a failure stress $\tau_A^f$, the applied load at which failure begins is reduced from the value in Eq. (19) by a factor of

$$1-\varepsilon(\beta+\kappa)/(2\beta+\kappa)+\mathcal{O}(\varepsilon^2). \tag{25}$$

However, this initial failure does not spread catastrophically: an increasing load is required to propagate the de-bond. In fact, with the assumptions of the present model, the applied load can be increased up to the value in Eq. (19), albeit with an increasingly long de-bond. This is because a debond of length $d$ merely shifts the interval over which load transfer occurs from $0 \leqslant x < \infty$ to $d \leqslant x < \infty$, so that effectively the minimum glue-line thickness has been increased from $1-\varepsilon$ to

$$1-\varepsilon e^{-\kappa d},$$

while the characteristic decay length $\kappa$ is unchanged. By the preceding analysis the failure load has therefore been increased relative to the initial value for no debond. Thus a bonded joint may be said to be damage tolerant with respect to constrictions in the adhesive layer because its strength is not reduced, provided that the overlap is sufficiently long. However, the work done by the applied load is increased if debonding occurs, and this is important in the repair context[4,5] because it implies that the crack extension force is increased.

With the choice of the plus sign in Eq. (23), the maximum value of $\tau_A$ is no longer necessarily at $x = 0$. The equation $\tau'_A(x) = 0$ derived from Eq. (24) admits a solution corresponding to a maximum in the range $0 < x < \infty$ if $\varepsilon > \beta/(\beta+\kappa)$. This imposes a restriction on the flaring of the adhesive layers: to ensure that the maximum $\tau_A$ occurs at $x = 0$ we must require that

$$\kappa/\beta < (1-\varepsilon)/\varepsilon. \tag{26}$$

Then, the applied load at which failure begins is increased from the value in Eq. (19) by a factor of

$$1+\varepsilon(\beta+\kappa)/(2\beta+\kappa)+\mathcal{O}(\varepsilon^2). \tag{27}$$

However, an initial failure will now spread catastrophically if the load is maintained.

The load-transfer length may be defined by the ratio $-\tau_A(0)/\tau'_A(0)$, and it is given by

$$(1/\beta)\{1 \pm \varepsilon(\beta+\kappa)^2/[\beta(2\beta+\kappa)]+\mathcal{O}(\varepsilon^2)\}, \tag{28}$$

showing that it is respectively increased or decreased by a flaring or a constriction, relative to the value $1/\beta$ which it would have for constant thickness.

There are two important limitations of the present analysis which must be kept in mind when applying the results derived above or further results obtained by the same procedure. First, the classical theory involves a physical inconsistency in predicting a non-zero value for the adhesive shear stress at the free end $x = 0$. This raises a problem of interpretation.[2] Detailed two-dimensional analyses have shown that the results of the one-dimensional theory are accurate except within a distance from the end $x = 0$ approximately equal to the thickness of the adhesive layer. Thus, the one-dimensional theory should be adequate provided that the characteristic lengths involved, namely $1/\beta$ and $1/\kappa$, are much larger than the nominal adhesive thickness $t_A$, which is usually the case in practice. Secondly, the accuracy of a perturbation expansion relies on the parameter $\varepsilon$ being sufficiently small. To assess the range of $\varepsilon$ for which the two-term expansion in Eq. (24) yields sufficiently accurate results for practical purposes, we shall later compare these results with those of an exact solution derived in the following section.

## INVERSE METHOD

One can derive an exact solution of Eq. (7) by specifying $\sigma_P(x)$ and solving Eq. (7) for $h(x)$. For this approach to be useful, the resulting glue-line thickness $t_A h(x)$ must be physically realisable, *i.e.* $h(x)$ must be positive and $\mathcal{O}(1)$ for all $x$. Further, for our purposes, $h(x)$ should display a constriction or flaring near

$x = 0$ and it should decay to a constant (non-zero) value for $x \to \infty$. Accordingly, let us choose $\sigma_P(x)$ in the form

$$\sigma_P(x) = -(F/t_P) \exp(-\beta x) g(x), \tag{29}$$

so that we can recover the classical case of constant glue-line thickness by specifying $g(x) = 1$, *cf.* Eq. (17). To satisfy Eq. (9) we must impose the boundary condition

$$g(x = 0) = 1. \tag{30}$$

Integrating Eq. (7) we obtain

$$\{e^{-\beta x} g(x)\}' h(x) = \beta^2 \int_0^x e^{-\beta t} g(t)\, dt + C, \tag{31}$$

where the constant of integration $C$ is determined by the observation that the left hand side decays to zero when $x \to \infty$ because of the exponential term. Thus Eq. (31) leads to

$$h(x) = -[\beta^2/\{e^{-\beta x} g(x)\}'] \int_x^\infty e^{-\beta t} g(t)\, dt. \tag{32}$$

At this stage one could experiment numerically with various specifications for $g(x)$. An analytically convenient choice, which satisfies Eq. (30), proves to be

$$g(x) = (1 + kx)^{-1}, \quad k > 0, \tag{33}$$

for which the adhesive shear stress derived from Eqs. (29) and (5) is

$$\tau_A(x) = F\{k + \beta(1 + kx)\}(1 + kx)^{-2} e^{-\beta x}, \tag{34}$$

and the corresponding thickness variation is given by

$$h(x) = \beta^2 (1 + kx)^2 \{k + \beta(1 + kx)\}^{-1} e^{\beta x} I(x), \tag{35}$$

with

$$I(x) \equiv \int_x^\infty (1 + kt)^{-1} e^{-\beta t}\, dt. \tag{36}$$

This $h(x)$ meets the consistency conditions $h(x) > 0$ for all $x$ and $h(x \to \infty) \to 1$, since

$$I(x \to \infty) = e^{-\beta x}\{(\beta k x)^{-1} + \mathcal{O}(x^{-2})\}. \tag{37}$$

To study the form of $h(x)$ near $x = 0$ we first note that

$$I(x \to 0) = I_0 - x + \tfrac{1}{2}(\beta + k)x^2 + \mathcal{O}(x^3), \tag{38}$$

where

$$I_0 \equiv k^{-1} e^{\beta/k} E_1(\beta/k) \tag{39}$$

and $E_1$ denotes the exponential integral, a tabulated function.[8] Thus Eq. (35) leads to

$$h(x \to 0) = h(0) + h'(0)x + \mathcal{O}(x^2), \tag{40}$$

with

$$h(0) = \beta^2(\beta+k)^{-1}I_0, \tag{41}$$

$$h'(0) = \beta^2(\beta+k)^{-2}\{(\beta^2+2\beta k+2k^2)I_0-(\beta+k)\}. \tag{42}$$

## Discussion

The choice of $g(x)$ in Eq. (33) leads to a constriction in the adhesive layer, the minimum thickness being given by $t_A h(0)$, as compared with $t_A(1-\varepsilon)$ in the perturbation method. The characteristic length of this constriction is given by

$$\{1-h(0)\}/h'(0), \tag{43}$$

which corresponds to the length $1/\kappa$ in the perturbation method. With the present method, the minimum thickness can be made arbitrarily small, as indicated by the values of $h(0)$ in Table I, so that one can assess the influence of quite pronounced constrictions for which one would not expect the two-term perturbation expansion to yield accurate results. However, the perturbation method is more flexible relative to the form of thickness variation which can be studied.

To compare the two methods, consider first the case $\beta/k = 10$ for which $h(0) = 0.832$ and $h'(0) = 0.014\beta$ from Table I. The corresponding values of the parameters used in the perturbation method are $\varepsilon = 0.168$ and $\kappa = 0.083\beta$. Then the exact value of $\tau_A(0)$ from Eq. (34) is

$$\tau_A(0)\ F\beta(1+k/\beta), \tag{44}$$

$$= 1.1\ F\beta,$$

TABLE I

Numerical values for Eq. (40)

| $\beta/k$ | $e^{\beta/k}E_1(\beta/k)$ | $h(0)$ | $h'(0)/\beta$ |
|---|---|---|---|
| 0.1 | 2.014 | 0.018 | 0.277 |
| 0.5 | 0.923 | 0.154 | 0.333 |
| 1 | 0.596 | 0.298 | 0.245 |
| 2 | 0.361 | 0.482 | 0.136 |
| 5 | 0.170 | 0.710 | 0.042 |
| 10 | 0.092 | 0.832 | 0.014 |

while the two-term expansion in Eq. (24) gives

$$\tau_A(0) = F\beta\{1+\varepsilon(\beta+\kappa)/(2\beta+\kappa)\}, \quad (45)$$
$$= 1.087\, F\beta,$$

*i.e.* the two results differ by only 1.3%. The same calculations repeated for the cases $\beta/k = 5$ and 2, for which $\varepsilon = 0.29$ and 0.52 and $\kappa/\beta = 0.145$ and 0.263 respectively, show a difference of 4% and 14% respectively. Thus we conclude that the perturbation expansion can be expected to give sufficiently accurate results for practical purposes (*i.e.* with an error no worse than 5%) for relative thickness variations up to approximately one quarter of the nominal thickness.

In the preceding comparison, the thickness variations have been described by two parameters, namely their *amplitude*, which gives the maximum relative thickness variation, and a characteristic *decay length*, which is determined from the slope at the point of maximum thickness variation. In terms of this two-parameter characterization, the results obtained by the inverse method can be applied to estimate the influence of constrictions without limitations on the amplitude of the constriction. Thus, if we define the load-transfer length as the ratio $-\tau_A(0)/\tau'_A(0)$, we derive the following formula for this transfer length from Eq. (34):

$$\text{transfer length} = (1+k/\beta)/\{\beta+2k(1+k/\beta)\}. \quad (46)$$

Eq. (46) reduces to the classical value $1/\beta$ when $k/\beta \to 0$, and it agrees with Eq. (28) for small $\varepsilon$. However, unlike Eq. (28), Eq. (46) is not restricted to small constrictions. Similarly Eq. (44) gives an estimate of the maximum adhesive shear stress which is not restricted to small constrictions.



## CONCLUSION

The influence of variations in glue-line thickness has been studied by two methods. The first, a perturbation method, is easy to apply and flexible with respect to the form of thickness variation which can be studied, but it is an asymptotic method which cannot be expected to be accurate unless the amplitude of variation is sufficiently small. A numerical comparison with the results of an exact solution obtained by an inverse method suggests that the two-term perturbation expansions of the type given in Eqs (24, 25, 27, 28, 45) should be sufficiently accurate in practice for thickness variations up to one quarter of the nominal thickness.

If the thickness variation is prescribed, the basic equation, Eq. (7), cannot in general be solved in closed form. However, using an inverse method in which the stress distribution is prescribed, one can derive a closed-form expression for the corresponding thickness variation. This expression includes an integral which cannot be reduced to elementary functions, but one can derive analytically the amplitude and the decay length of the thickness variation. The results obtained by this inverse method, for example Eqs (34, 44, 46), are not restricted to small amplitudes. Furthermore, the present one-dimensional results can give an indication of the effects of thickness variations in more general contexts, for example where a two-dimensional analysis would strictly be required.

## References

1. I. N. Sneddon, in *Adhesion*, D. D. Eley, ed. (Oxford University Press, 1961).
2. L. J. Hart-Smith, *Analysis and Design of Advanced Composite Bonded Joints*, NASA CR-2218 (1974).
3. L. J. Hart-Smith, Further Developments in the Design and Analysis of Adhesive-Bonded Structural Joints, *ASTM Symposium on Joining of Composite Materials* (1980).
4. L. R. F. Rose, A Cracked Plate Repaired by Bonded Reinforcements, *Int. J. Fracture* (in press, 1981).
5. A. A. Baker, J. D. Roberts and L. R. F. Rose, Use of Overlap Joint Parameters in Estimating the K Reduction due to Crack-Patching, *Proc. International Workshop on Defence Applications of Advanced Repair Technology for Metal and Composite Structures*, Naval Research Labs., Washington, DC (1981).
6. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 9.
7. E. H. Mansfield, Inverse Methods in Two-dimensional Elasticity, *Proc. 13th International Congress of Theoretical and Applied Mechanics*, E. Becker and G. K. Mikhailov, eds. (Springer-Verlag, Berlin, 1973).
8. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions* (Dover Publications, New York, 1965), Chap. 5.